

Riemann-Cartan Space-times of Gödel Type

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Abstract

A class of Riemann-Cartan Gödel-type space-times are examined in the light of the equivalence problem techniques. The conditions for local space-time homogeneity are derived, generalizing previous works on Riemannian Gödel-type space-times. The equivalence of Riemann-Cartan Gödel-type space-times of this class is studied. It is shown that they admit a five-dimensional group of affine-isometries and are characterized by three essential parameters ℓ, m^2, ω : identical triads (ℓ, m^2, ω) correspond to locally equivalent manifolds. The algebraic types of the irreducible parts of the curvature and torsion tensors are also presented.

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1 Introduction

Theories with non-zero torsion have been of notable interest in a few contexts. In the framework of gauge theories they have been used in the search for unification of gravity with the other fundamental interactions in physics. Space-time manifolds with nonsymmetric connection have also been considered as the appropriate arena for the formulation of a quantum gravity theory. At a classical level the generalization of the standard general relativity by introducing torsion into the theory has also received a good deal of attention mainly since the sixties (see Hehl *et al.* [1] and references therein). The geometric concept of torsion has also been used in a continuum approach to lattice defects in solids since the early fifties [2] – [6]. For further motivation and physical consequences of studying manifolds and theories with non-zero torsion (whether dropping the metricity condition or not) as well as for a detailed list of references on theories with non-zero torsion, we refer the readers to a recent review by Hehl *et al.* [7].

In general relativity (GR), the space-time M is a four-dimensional Riemannian manifold endowed with a locally Lorentzian metric and a metric-compatible symmetric connection, namely the Christoffel symbols $\{\Gamma^a_{bc}\}$. However, it is well known that the metric tensor and the connection can be introduced as independent structures on a given space-time manifold M .

In GR there is a unique torsion-free connection on M . In the framework of torsion theories of gravitation (TTG), on the other hand, we have Riemann-Cartan (RC) manifolds, i.e., space-time manifolds endowed with locally Lorentzian metrics and metric-compatible nonsymmetric connections Γ^a_{bc} . Thus, in TTG the connection has a metric-independent part given by the torsion, and for a characterization of the local gravitational field, one has to deal with both metric and connection.

The arbitrariness in the choice of coordinates is a basic assumption in GR and in TTG. Nevertheless, in these theories it gives rise to the problem of deciding whether

or not two apparently different space-time solutions of the field equations are locally the same — the equivalence problem. In GR this problem can be couched in terms of local isometry, whereas in TTG besides local isometry ($g_{ab} \rightarrow \tilde{g}_{ab}$) it means affine collineation ($\Gamma^a_{bc} \rightarrow \tilde{\Gamma}^a_{bc}$) of two RC manifolds.

The equivalence problem in general relativity (Riemannian space-times) has been discussed by several authors and is of interest in many contexts (see, for example, Cartan [8], Karlhede [9], MacCallum [10] – [12] and references therein).

The equivalence problem in torsion theories of gravitation (RC space-times), on the other hand, was only discussed recently [13]. Subsequently, an algorithm for checking the equivalence in TTG and a first working version of a computer algebra package (called TCLASSI) which implements this algorithm have been presented [14] – [16].

The Gödel [17] solution of Einstein’s field equations is a particular case of the Gödel-type line element, defined by

$$ds^2 = [dt + H(x) dy]^2 - D^2(x) dy^2 - dx^2 - dz^2, \quad (1.1)$$

in which

$$H(x) = e^{mx}, \quad D(x) = e^{mx}/\sqrt{2}, \quad (1.2)$$

and with the energy-momentum tensor $T_{\mu\nu}$ given by

$$T_{\mu\nu} = \rho v_\mu v_\nu, \quad v^\alpha = \delta^\alpha_0, \quad (1.3)$$

$$\kappa\rho = -2\Lambda = m^2 = 2\omega^2, \quad (1.4)$$

where κ and Λ are, respectively, the Einstein gravitational and the cosmological constants, ρ is the fluid density and v^α its four-velocity, and ω is the rotation of the matter. The Gödel model is homogeneous in space-time (hereafter called ST homogeneous). Actually it admits a five parameter group of isometries (G_5) having an isotropy subgroup of dimension one (H_1).

The problem of space-time homogeneity of four-dimensional Riemannian manifolds endowed with a Gödel-type metric (1.1) was considered for the first time by Raychaudhuri and Thakurta [18]. They have determined the necessary conditions for space-time homogeneity. Afterwards, Rebouças and Tiomno [19] proved that the Raychaudhuri-Thakurta necessary conditions are also sufficient for ST homogeneity of Gödel-type Riemannian space-time manifolds. However, in both articles [18, 19] the study of ST homogeneity is limited in that only time-independent Killing vector fields were considered [20]. The necessary and sufficient conditions for a Gödel-type Riemannian space-time manifold to be ST homogeneous were finally rederived without assuming such a simplifying hypothesis in [21], where the equivalence problem techniques for Riemannian space-times, as formulated by Karlhede [9] and implemented in CLASSI [22], were used.

In this article, in the light of the equivalence problem techniques for Riemann-Cartan space-times, as formulated by Fonseca-Neto *et al.* [13] and embodied in the suite of computer algebra programs TCLASSI [14] – [16], we shall examine all Riemann-Cartan manifolds endowed with a Gödel-type metric (1.1), with a torsion polarized along the preferred direction defined by the rotation, and sharing the translational symmetries of $g_{\mu\nu}$ in (1.1). Hereafter, for the sake of brevity, we shall refer to this family of space-time manifolds as Riemann-Cartan Gödel-type manifolds. The necessary and sufficient conditions for a Riemann-Cartan Gödel-type manifold to be ST (locally) homogeneous are derived. The Åman-Rebouças results [21] for Riemannian Gödel-type space-times are generalized. The ST homogeneous Riemann-Cartan Gödel-type manifolds are shown to admit a five-dimensional group of affine-isometric motions. The equivalence of these Riemann-Cartan space-times is discussed: they are found to be characterized by three essential parameters m^2 , ω and ℓ : identical triads (ℓ, m^2, ω) correspond to equivalent manifolds. The algebraic classification of the nonvanishing irreducible parts of the curvature is presented. For a general triad (ℓ, m^2, ω) , the Weyl-type spinors Ψ_A and ψ_A are both Petrov type D, while the non-null Ricci-type spinors $\Phi_{AB'}$ and $\phi_{AB'}$ are both Segre type [1,1(11)]. A few main

instances for which these algebraic types can be more specialized are also studied. The classification of the irreducible parts of the torsion and the corresponding group of isotropy are also discussed. The pseudo-trace torsion spinor $\mathcal{P}_{AX'}$ is found to be space-like, with $SO(2,1)$ as its group of isotropy. The Lanczos spinor $\mathcal{L}_{ABCX'}$ is found to be invariant under one-dimensional spatial rotation.

Our major aim in the next section is to present a brief summary of some important theoretical and practical results on the equivalence problem for Riemann-Cartan space-times required in Section 3, where we state, prove and discuss our main results.

2 Equivalence of Riemann-Cartan Space-times: Basic Results

Most relativists's first approach to solving the (local) equivalence problem of Riemann-Cartan manifolds would probably be to make use of the so-called scalar polynomial invariants built from the curvature, the torsion, and their covariant derivatives [23]. However, this attempt cannot work since there exist *curved* plane wave RC space-times with non-zero torsion [24] for which all the scalar polynomial invariants vanish — indistinguishable therefore from the Minkowski space (flat and torsion free). This example shows that although necessary the scalar polynomial invariants are not sufficient to distinguish (locally) two RC space-times.

To make apparent that the conditions for the local equivalence of RC manifolds follow from Cartan's approach to the equivalence problem, we shall first recall the definition of equivalence and then proceed by pointing out how Cartan's results [8] lead to the solution of the problem found in [13]. The basic idea is that if two Riemann-Cartan manifolds M and \widetilde{M} are the same, they will define identical Lorentz frame bundles [$L(M) \equiv L(\widetilde{M})$]. The manifold $L(M)$ incorporates the freedom in the choice of Lorentz frames and has a uniquely-defined set of linearly independent 1-forms $\{\Theta^A, \omega^A_B\}$, forming a basis of the

cotangent space $T_P^*(L(M))$ at an arbitrary point $P \in L(M)$. Two RC manifolds M and \widetilde{M} are then said to be locally equivalent when there exists a local mapping F between the Lorentz frame bundles $L(M)$ and $L(\widetilde{M})$ such that (see [13] and also Ehlers [25])

$$F^* \widetilde{\Theta}^A = \Theta^A \quad \text{and} \quad F^* \widetilde{\omega}_B^A = \omega_B^A \quad (2.1)$$

hold. Here F^* is the well known pull-back map defined by F .

A solution to the equivalence problem for Riemann-Cartan manifolds can then be obtained by using Cartan's results on the equivalence of sets of 1-forms (see p. 312 of the English translation of Ref. [8]) together with Cartan equations of structure for a manifold endowed with a nonsymmetric connection. The solution can be summarized as follows [13, 16].

Two n -dimensional Riemann-Cartan (locally Lorentzian) manifolds M and \widetilde{M} are locally equivalent if there exists a local map (diffeomorphism) F between their corresponding Lorentz frame bundles $L(M)$ and $L(\widetilde{M})$, such that the *algebraic* equations relating the components of the curvature and torsion tensors and their covariant derivatives:

$$\begin{aligned} T_{BC}^A &= \widetilde{T}_{BC}^A, \\ R_{BCD}^A &= \widetilde{R}_{BCD}^A, \\ T_{BC;M_1}^A &= \widetilde{T}_{BC;M_1}^A, \\ R_{BCD;M_1}^A &= \widetilde{R}_{BCD;M_1}^A, \\ T_{BC;M_1 M_2}^A &= \widetilde{T}_{BC;M_1 M_2}^A, \\ &\vdots \\ R_{BCD;M_1 \dots M_{p+1}}^A &= \widetilde{R}_{BCD;M_1 \dots M_{p+1}}^A, \\ T_{BC;M_1 \dots M_{p+2}}^A &= \widetilde{T}_{BC;M_1 \dots M_{p+2}}^A \end{aligned} \quad (2.2)$$

are compatible as equations in Lorentz frame bundle coordinates (x^a, ξ^A) . Here and in what follows we use a semicolon to denote covariant derivatives. Note that x^a are

coordinates on the manifold M while ξ^A parametrize the group of allowed frame transformations. Reciprocally, equations (2.2) imply local equivalence between the space-time manifolds. The $(p+2)^{th}$ derivative of torsion and the $(p+1)^{th}$ derivative of curvature are the lowest derivatives which are functionally dependent on all the previous derivatives. It should be noticed that in the above set of *algebraic* equations *necessary and sufficient* for the local equivalence we have taken into account the Bianchi identities $R^A_{[BCD]} - T^A_{[BC;D]} = -T^N_{[BC}T^A_{D]N}$ and their differential concomitants. Thus, when the components of the $0^{th}, \dots, (p+1)^{th}$ covariant derivatives of the torsion are known, the Bianchi identities and their differential concomitants reduce to a set of linear algebraic equations, which relates (for each p) the $(p+1)^{th}$ covariant derivatives of curvature to the $(p+2)^{th}$ covariant derivatives of torsion. So we need the $(p+2)^{th}$ derivatives of torsion in (2.2), which did not appear in [13].

A comprehensive local description of a Riemann-Cartan manifold is, therefore, given by the set

$$I_p = \{T^A_{BC}, R^A_{BCD}, T^A_{BC;N_1}, R^A_{BCD;M_1}, T^A_{BC;N_1N_2}, \dots, R^A_{BCD;M_1\dots M_p}, T^A_{BC;N_1\dots N_{p+1}}\}, \quad (2.3)$$

whose elements are called Cartan scalars, since they are scalars under coordinate transformations on the base manifold. The theoretical upper bound for the number of covariant derivatives to be calculated is 10 for the curvature and 11 for the torsion, which corresponds to 11 steps (from 0th to 10th-order derivatives for the curvature) in the algorithm presented below. The number of steps can be thought of as being related to the six Lorentz transformation parameters ξ^A , the four coordinates x^a on the space-time manifold and one integrability condition. A word of clarification regarding this integrability condition is in order here: when the number of derivatives needed is not the maximum possible (set by the dimension of the frame bundle) then to show that the derivative process has terminated one has to take one more derivative and show that it contains no

new information by checking the functional relations between the Cartan scalars. This can be understood as if we were introducing invariantly-defined coordinates (though we cannot explicitly do that) and then had to take their derivatives in order to substitute for the differentials in the usual formula for the line element. In practice, the coordinates and Lorentz transformation parameters are treated differently. Actually a fixed frame (a local section of the Lorentz frame bundle) is chosen to perform the calculations so that the elements of the set I_p coincide with the components of the curvature and torsion tensors of the space-time base manifold and their covariant derivatives; there is no explicit dependence on the Lorentz parameters.

To deal with equivalence it is necessary to calculate the elements of the set I_p . However, even when the Bianchi and Ricci identities and their differential concomitants are taken into account, in the worst case one still has 11064 independent elements to calculate. Thus, an algorithmic procedure for carrying out these calculations and a computer algebra implementation are highly desirable.

A practical procedure for testing equivalence of Riemann-Cartan space-times has been developed [14] – [16], [26]. In the procedure the maximum order of derivatives is not more than 7 for the curvature and 8 for the torsion. The basic idea behind our procedure is separate handling of frame rotations and space-time coordinates, fixing the frame at each stage of differentiation of the curvature and torsion tensors by aligning the basis vectors as far as possible with invariantly-defined directions. The algorithm starts by setting $q = 0$ and has the following steps [16]:

1. Calculate the set I_q , i.e., the derivatives of the curvature up to the q^{th} order and of the torsion up to the $(q + 1)^{th}$ order.
2. Fix the frame, as much as possible, by putting the elements of I_q into canonical forms.
3. Find the frame freedom given by the residual isotropy group H_q of transfor-

mations which leave the canonical forms invariant.

4. Find the number t_q of functionally independent functions of space-time coordinates in the elements of I_q , brought into the canonical forms.
5. If the isotropy group H_q is the same as $H_{(q-1)}$ and the number of functionally independent functions t_q is equal to $t_{(q-1)}$, then let $q = p + 1$ and stop. Otherwise, increment q by 1 and go to step 1.

This procedure provides a discrete characterization of Riemann-Cartan space-times in terms of the following properties: the set of canonical forms in I_p , the isotropy groups $\{H_0, \dots, H_p\}$ and the number of independent functions $\{t_0, \dots, t_p\}$. Since there are t_p essential space-time coordinates, clearly $4 - t_p$ are ignorable, so the isotropy group will have dimension $s = \dim(H_p)$, and the group of symmetries (called affine-isometry) of both metric (isometry) and torsion (affine collineation) will have dimension r given by

$$r = s + 4 - t_p, \quad (2.4)$$

acting on an orbit with dimension

$$d = r - s = 4 - t_p. \quad (2.5)$$

To check the equivalence of two Riemann-Cartan space-times one first compares the above discrete properties and only when they match is it necessary to determine the compatibility of equations (2.2).

In our implementation of the above practical procedure, rather than using the curvature and torsion tensors as such, the algorithms and computer algebra programs were devised and written in terms of spinor equivalents: (i) the irreducible parts of the Riemann-Cartan curvature spinor

$$\begin{aligned} R_{ABCDG'H'} &= \varepsilon_{G'H'} [\Psi_{ABCD} + (\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC})(\Lambda + i\Omega) + \varepsilon_{AC}\Sigma_{BD} \\ &+ \varepsilon_{BD}\Sigma_{AC} + \varepsilon_{AD}\Sigma_{BC} + \varepsilon_{BC}\Sigma_{AD}] + \varepsilon_{CD}(\Phi_{ABG'H'} + i\Theta_{ABG'H'}), \end{aligned} \quad (2.6)$$

which, clearly, are Ψ_{ABCD} (TPSI), $\Phi_{ABX'Z'}$ (TPHI), $\Theta_{ABX'Z'}$ (THETA), Σ_{AB} (SIGMA), Λ (TLAMBD) and Ω (OMEGA); and (ii) the irreducible parts of the torsion spinor

$$T_{AX'BC} = L_{X'ABC} + \frac{1}{3}(\varepsilon_{AB}T_{CX'} + \varepsilon_{AC}\bar{T}_{BX'}) + \frac{1}{3}i(\varepsilon_{AB}S_{CX'} + \varepsilon_{AC}\bar{S}_{BX'}), \quad (2.7)$$

namely: $\mathcal{T}_{AX'}$ (SPTTOR, SP = spinor, T = trace, TOR = torsion), $\mathcal{P}_{AX'}$ (SPPTOR, SP = spinor, P = pseudo-trace, TOR = torsion), and $\mathcal{L}_{ABCX'}$ (SPLTOR, SP = spinor, L = Lanczos spinor, TOR = torsion). Note that these irreducible parts of curvature and torsion spinors are nothing but the spinor equivalents of the curvature and torsion tensors given, respectively, by equations (B.4.3) and (B.2.5) in the appendix B of Ref. [7]. Note that the TCLASSI users' names for the spinorial quantities have been indicated inside round brackets. We note that, besides the above indication for the names of the irreducible parts of the torsion spinor, the names of the irreducible parts of both Riemann-Cartan curvature and first covariant derivative of the torsion were generalized from the names in CLASSI [12] by bearing in mind whether they have the same symmetry as the Weyl spinor (the Weyl-type spinors: Ψ_A and ψ_A) or the symmetry of the Ricci spinor (the Ricci-type spinors: $\Phi_{AB'}$, $\phi_{AB'}$, $\Theta_{AB'}$, $\nabla\mathcal{T}_{AX'}$, $\nabla\mathcal{P}_{AY'}$). We have employed the affixes: BV for bivector, SP for spinor, SC for scalar, A for d'Alembertian; D, D2 and so on, for the first, the second and so forth derivative of the spinorial quantities. We have used the letters Σ , \mathcal{M} , \mathcal{B} to denote bivectors, i.e., objects with the same symmetries of Maxwell spinor. We have also named three basic scalars by TLAMBD (Λ) OMEGA (Ω) and SCTTOR (\mathcal{T}). There are also names which were simply borrowed from CLASSI with the addition of the letter T for torsion, as for example TXI (see [12], [14] and [16] for details).

A relevant point to be taken into account when one needs to compute derivatives of the curvature and the torsion tensors is that they are interrelated by the Bianchi and Ricci identities and their differential concomitants. Thus, to cut down the number of quantities to be calculated it is very important to find a minimal set of quantities from which the curvature and torsion tensors, and their covariant derivatives are obtainable by algebraic

operations. For Riemann-Cartan space-time manifolds, taking into account the irreducible parts of the Bianchi and Ricci identities and their differential concomitants, a complete minimal set C_q of such quantities recursively defined in terms of totally symmetrized q^{th} derivatives of the curvature spinors and $(q+1)^{th}$ derivatives of the torsion spinors can be specified [16, 27] by:

1. For $q = 0$: the torsion's irreducible parts, namely
 - (a) $\mathcal{T}_{AX'}$ (SPTTOR), (b) $\mathcal{P}_{AX'}$ (SPPTOR), (c) $\mathcal{L}_{ABCX'}$ (SPLTOR);
2. The totally symmetrized q^{th} derivatives of
 - (a)
 - i. Ψ_{ABCD} (TPSI),
 - ii. $\psi_{ABCD} \equiv -\nabla^{N'}_{(A}\mathcal{L}_{BCD)N'}$ (PSILTORT),
 - (b)
 - i. $\Phi_{ABX'Z'}$ (TPHI),
 - ii. $\Theta_{ABX'Z'}$ (THETA),
 - iii. $\phi_{ABX'Z'} \equiv -\frac{1}{2}(\nabla^N_{(X'}\mathcal{L}_{Z')ABN} + \nabla^{N'}_{(A}\bar{\mathcal{L}}_{B)X'Z'N'})$ (PHILTORT),
 - (c)
 - i. Λ (TLAMBD),
 - ii. Ω (OMEGA),
 - iii. $\mathcal{T} \equiv \nabla_{NN'}\mathcal{T}^{NN'}$ (SCTTOR),
 - (d)
 - i. Σ_{AB} (SIGMA),
 - ii. $\mathcal{M}_{AB} \equiv \nabla^{N'}_{(A}\mathcal{T}_{B)N'}$ (BVTTOR),
 - iii. $\mathcal{B}_{AB} \equiv \nabla^{N'}_{(A}\mathcal{P}_{B)N'}$ (BVPTORT).
3. The totally symmetrized $(q+1)^{th}$ derivatives of (a) $\mathcal{T}_{AX'}$ (DSPTTOR), (b) $\mathcal{P}_{AX'}$ (DSPPTORT), (c) $\mathcal{L}_{ABCX'}$ (DSPLTOR).
4. For $q \geq 1$:
 - (a) the totally symmetrized $(q-1)^{th}$ derivatives of

- i. $\Xi_{ABCX'} \equiv \nabla_{X'}^N \Psi_{ABCN}$ (TXI),
 - ii. $\mathcal{X}_{ABCX'} \equiv \nabla_{(A}^{N'} \Theta_{BC)N'X'}$ (XITH),
 - iii. $\mathcal{U}_{AX'} \equiv \frac{1}{2}(\nabla_{X'}^N \Sigma_{AN} + \nabla_A^{N'} \bar{\Sigma}_{X'N'})$ (TSIGM),
 - iv. $\mathcal{V}_{AX'} \equiv -\frac{i}{2}(\nabla_{X'}^N \Sigma_{AN} - \nabla_A^{N'} \bar{\Sigma}_{X'N'})$ (PSIGM);
- (b) for $q = 1$ the d'Alembertian of the irreducible parts of the torsion:
- i. $\square \mathcal{T}_{AX'} \equiv \nabla^{NN'} \nabla_{NN'} \mathcal{T}_{AX'}$ (ASPTTOR),
 - ii. $\square \mathcal{P}_{AX'} \equiv \nabla^{NN'} \nabla_{NN'} \mathcal{P}_{AX'}$ (ASPPTOR),
 - iii. $\square \mathcal{L}_{ABCX'} \equiv \nabla^{NN'} \nabla_{NN'} \mathcal{L}_{ABCX'}$ (ASPLTOR).

5. For $q \geq 2$:

- (a) the d'Alembertian $\square Q \equiv \nabla^{NN'} \nabla_{NN'} Q$ applied to all quantities Q calculated for the derivatives of order $q-2$, i.e. in the set $C_{(q-2)}$, except the d'Alembertians of the irreducible parts of torsion for $q = 2$ (when $n = 2$, e.g., $\square \Psi_A$ (ATPSI), $\square \psi_A$ (APSILTOR), $\square \Phi_{AB'}$ (ATPHI), and so forth).
- (b) the totally symmetrized $(q-2)^{th}$ derivatives of
- i. $\Upsilon_{ABCD} \equiv -\nabla_{(A}^{N'} \mathcal{X}_{BCD)N'}$ (PSIXITH),
 - ii. $\mathcal{F}_{AB} \equiv \nabla_{(A}^{N'} \mathcal{U}_{B)N'}$ (BVTSIGM).

It should be stressed that we have included in the above set the d'Alembertian of the irreducible parts of the torsion (4.(b) i – iii), which was missed in [14]. Note also that the above list contains inside parentheses the TCLASSI external name (for the users) after each quantity. Finally, we remark that the above minimal set is a generalization of the corresponding set found for Riemannian space-time manifolds by MacCallum and Åman [28].

In our practical procedure the frame is fixed (as much as possible) by bringing into canonical form first the quantities with the same symmetry as the Weyl spinor (called

Weyl-type), i.e., Ψ_A and ψ_A , followed by the spinors with the symmetry of the Ricci spinor (referred to as Ricci-type spinor), namely $\Phi_{AB'}$, $\phi_{AB'}$, $\Theta_{AB'}$, $\nabla\mathcal{T}_{AX'}$, $\nabla\mathcal{P}_{AY'}$, then bivector spinors Σ_{AB} , \mathcal{M}_{AB} and \mathcal{B}_{AB} , and finally vectors $\mathcal{T}_{AX'}$ and $\mathcal{P}_{AX'}$ are taken into account. Thus, if Ψ_A is Petrov I, for example, the frame can be fixed by demanding that the nonvanishing components of Ψ_A are such that $\Psi_1 = \Psi_3 \neq 0, \Psi_2 \neq 0$. Clearly an alternative canonical frame is obtained by imposing $\Psi_0 = \Psi_4 \neq 0, \Psi_2 \neq 0$. Although the latter is implemented in TCLASSI as the canonical frame, in the next section we shall use the former (defined to be an acceptable alternative in TCLASSI) to make easier the comparison between our results and those of the corresponding Riemannian case [21].

To close this section we remark that in the TCLASSI implementation of the above results a notation is used in which the indices are all subscripts and components are labelled by a primed and unprimed index whose numerical values are the sum of corresponding (primed and unprimed) spinor indices. Thus, for example, one has

$$\nabla\Psi_{20'} \equiv \Psi_{(1000;1)0'} = \nabla^{X'}{}_{(A}\Psi_{BCDE)}\iota^A\iota^B o^C o^D o^E \bar{o}_{X'}, \quad (2.8)$$

where the parentheses indicate symmetrization, the bar is used for complex conjugation, and the pair (ι^A, o^B) constitutes an orthonormal spinor basis.

3 Homogeneous Riemann-Cartan Gödel-type Space-times

Throughout this section we shall consider a four-dimensional Riemann-Cartan manifold M , endowed with a Gödel-type metric (1.1) and a torsion that shares the same translational symmetries as the metric, and is aligned with the direction singled out by the rotation vector field w (called Riemann-Cartan Gödel-type space-time). So, in the coordinate system given in (1.1) the torsion tensor is given by $T^t{}_{xy} = S(x)$.

For arbitrary functions $H(x)$, $D(x)$ and $S(x)$ both Ψ_A and ψ_A are Petrov I; this fact

can be easily checked by using the package TCLASSI. Accordingly the null tetrad Θ^A which turns out to be appropriate (canonical) for our discussions is

$$\begin{aligned}\Theta^0 &= \frac{1}{\sqrt{2}}(\theta^0 + \theta^3), & \Theta^1 &= \frac{1}{\sqrt{2}}(\theta^0 - \theta^3), \\ \Theta^2 &= \frac{1}{\sqrt{2}}(\theta^2 - i\theta^1), & \Theta^3 &= \frac{1}{\sqrt{2}}(\theta^2 + i\theta^1),\end{aligned}\tag{3.1}$$

where θ^A is a Lorentz tetrad ($\eta_{AB} = \text{diag}(+1, -1, -1, -1)$) given by

$$\theta^0 = dt + H(x) dy, \quad \theta^1 = dx, \quad \theta^2 = D(x) dy, \quad \theta^3 = dz.\tag{3.2}$$

Clearly in the null frame (3.1) the torsion tensor and the Gödel-type line element (1.1) are given by

$$T_{23}^0 = T_{23}^1 = \frac{\sqrt{2}}{2} i S(x) \quad \text{and} \quad ds^2 = 2(\Theta^0 \Theta^1 - \Theta^2 \Theta^3).\tag{3.3}$$

It is worth mentioning that the Petrov type for Ψ_A and ψ_A and the canonical frame (3.1) were obtained by interaction with TCLASSI, starting from the Lorentz frame (3.2), changing to a null tetrad frame, and making dyad transformations to bring Ψ_A and ψ_A into the canonical form for Petrov type I discussed in section 2.

Using the TCLASSI package we referred to in the previous sections one finds the following nonvanishing components of the Cartan scalars corresponding to the first step (for $q = 0$) of our algorithm:

$$\Psi_1 = \Psi_3 = \frac{1}{8} \left[S' - \left(\frac{H'}{D} \right)' \right],\tag{3.4}$$

$$\Psi_2 = -\frac{S}{4} \left(\frac{S}{3} - \frac{H'}{D} \right) + \frac{1}{6} \left[\frac{D''}{D} - \left(\frac{H'}{D} \right)^2 \right],\tag{3.5}$$

$$\psi_1 = \psi_3 = \frac{S'}{8},\tag{3.6}$$

$$\psi_2 = -\frac{S}{4} \left(S - \frac{H'}{D} \right),\tag{3.7}$$

$$\Phi_{00'} = \Phi_{22'} = \frac{S}{4} \left(\frac{S}{2} - \frac{H'}{D} \right) + \frac{1}{8} \left(\frac{H'}{D} \right)^2,\tag{3.8}$$

$$\Phi_{01'} = \Phi_{12'} = -\frac{S'}{8} + \frac{1}{8} \left(\frac{H'}{D} \right)', \quad (3.9)$$

$$\Phi_{11'} = \frac{S}{4} \left(\frac{S}{4} - \frac{H'}{D} \right) + \frac{1}{4} \left[\frac{3}{4} \left(\frac{H'}{D} \right)^2 - \frac{D''}{D} \right], \quad (3.10)$$

$$\phi_{00'} = \phi_{22'} = \phi_{11'} = \frac{S}{4} \left(S - \frac{H'}{D} \right), \quad (3.11)$$

$$\phi_{01'} = \phi_{12'} = -\frac{S'}{8}, \quad (3.12)$$

$$\nabla \mathcal{P}_{01'} = -\nabla \mathcal{P}_{12'} = -\frac{i}{4} S', \quad (3.13)$$

$$\mathcal{B}_0 = -\mathcal{B}_2 = -\frac{i}{2} S', \quad (3.14)$$

$$\mathcal{P}_{00'} = -\mathcal{P}_{11'} = -\frac{\sqrt{2}}{2} S, \quad (3.15)$$

$$\Lambda = -\frac{S^2}{48} - \frac{1}{12} \left[\frac{D''}{D} - \frac{1}{4} \left(\frac{H'}{D} \right)^2 \right], \quad (3.16)$$

$$\mathcal{L}_{10'} = \mathcal{L}_{21'} = -\frac{i}{6} \sqrt{2} S, \quad (3.17)$$

$$\nabla \mathcal{L}_{10'} = -\nabla \mathcal{L}_{32'} = \frac{S}{16} \left(S - \frac{H'}{D} \right), \quad (3.18)$$

$$\nabla \mathcal{L}_{11'} = -\nabla \mathcal{L}_{31'} = -\frac{3}{4} \nabla \mathcal{L}_{20'} = \frac{3}{4} \nabla \mathcal{L}_{22'} = \frac{S'}{16}, \quad (3.19)$$

where the prime denotes derivative with respect to x .

From equation (2.5) one finds that for ST homogeneity we must have $t_p = 0$, that is the number of functionally independent functions of the space-time coordinates in the set I_p must be zero. Accordingly all the above quantities of the minimal set must be constant. Thus, from eqs. (3.4) – (3.19) one easily concludes that for a Riemann-Cartan Gödel-type space-time (3.3) to be ST homogeneous it is necessary that

$$S = \text{const} \equiv \ell, \quad (3.20)$$

$$\frac{H'}{D} = \text{const} \equiv 2\omega, \quad (3.21)$$

$$\frac{D''}{D} = \text{const} \equiv m^2. \quad (3.22)$$

We shall now show that the above necessary conditions are also sufficient for ST homogeneity. Indeed, under the conditions (3.20) – (3.22) the Cartan scalars corresponding

to the first step (for $q = 0$) of our algorithm reduce to

$$\Psi_2 = \frac{\ell}{2} \left(\omega - \frac{\ell}{6} \right) + \frac{m^2}{6} - \frac{2}{3} \omega^2, \quad (3.23)$$

$$\psi_2 = -\frac{\ell}{4} (\ell - 2\omega), \quad (3.24)$$

$$\Phi_{00'} = \Phi_{22'} = \frac{\ell}{4} \left(\frac{\ell}{2} - 2\omega \right) + \frac{\omega^2}{2}, \quad (3.25)$$

$$\Phi_{11'} = \frac{\ell}{4} \left(\frac{\ell}{4} - 2\omega \right) + \frac{3}{4} \omega^2 - \frac{m^2}{4}, \quad (3.26)$$

$$\phi_{00'} = \phi_{22'} = \phi_{11'} = \frac{\ell}{4} (\ell - 2\omega), \quad (3.27)$$

$$\mathcal{P}_{00'} = -\mathcal{P}_{11'} = -\frac{\sqrt{2}}{2} \ell, \quad (3.28)$$

$$\Lambda = -\frac{\ell^2}{48} + \frac{1}{12} (\omega^2 - m^2), \quad (3.29)$$

$$\mathcal{L}_{10'} = \mathcal{L}_{21'} = -\frac{i}{6} \sqrt{2} \ell, \quad (3.30)$$

$$\nabla \mathcal{L}_{10'} = -\nabla \mathcal{L}_{32'} = \frac{\ell}{16} (\ell - 2\omega). \quad (3.31)$$

Following the algorithm of the previous section, one needs to find the isotropy group which leaves the above Cartan scalars (canonical forms) invariant. Since $\ell \neq 0$ one can easily find that there are Cartan scalars invariant under the three-dimensional Lorentz group $SO(2, 1)$ like, e.g., $\mathcal{P}_{AB'}$, or even the whole Lorentz group, like Λ . However, the whole set of Cartan scalars (3.23) – (3.31) is invariant only under the spatial rotation

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \quad (3.32)$$

where α is a real parameter. So, the residual group H_0 which leaves the above Cartan scalars invariant is one-dimensional.

We proceed by carrying out the next step of our practical procedure, i.e., by calculating the totally symmetrized covariant derivative of the Cartan scalars (3.23) – (3.31) and the d'Alembertian of the irreducible parts of the torsion. Using TCLASSI one finds the

following nonvanishing quantities:

$$\begin{aligned}\nabla \Psi_{20'} &= -\nabla \Psi_{31'} = \frac{i}{40} \sqrt{2} \ell (2m^2 + 8\ell\omega - \ell^2 - 20\omega^2) \\ &\quad + \frac{i}{10} \sqrt{2} \omega (4\omega^2 - m^2),\end{aligned}\tag{3.33}$$

$$\nabla \psi_{20'} = -\nabla \psi_{31'} = -\frac{i}{40} 3 \sqrt{2} \ell (4\omega^2 - 4\ell\omega + \ell^2),\tag{3.34}$$

$$\begin{aligned}\Xi_{10'} &= \Xi_{21'} = \frac{i}{16} \sqrt{2} \ell (2m^2 + 8\ell\omega - \ell^2 - 20\omega^2) \\ &\quad + \frac{i}{4} \sqrt{2} \omega (4\omega^2 - m^2),\end{aligned}\tag{3.35}$$

$$\square \mathcal{L}_{10'} = \square \mathcal{L}_{21'} = \frac{i}{4} \sqrt{2} \ell (4\omega^2 - 4\ell\omega + \ell^2),\tag{3.36}$$

$$\nabla^2 \mathcal{L}_{10'} = \nabla^2 \mathcal{L}_{43'} = \frac{i}{80} \sqrt{2} \ell (4\omega^2 - 4\ell\omega + \ell^2),\tag{3.37}$$

$$\nabla^2 \mathcal{L}_{21'} = \nabla^2 \mathcal{L}_{32'} = -\frac{i}{480} \sqrt{2} \ell (4\omega^2 - 4\ell\omega + \ell^2).\tag{3.38}$$

As no new functionally independent function arose, $t_0 = t_1$. Besides, the Cartan scalars (3.33) – (3.38) are invariant under the same isotropy group (3.32), i.e. $H_0 = H_1$. Thus no new covariant derivatives should be calculated. From eq. (2.4) one finds that the group of symmetries (affine-isometric motions) of the Riemann-Cartan Gödel-type space-time is five-dimensional — the necessary conditions (3.20) – (3.22) are also sufficient for ST homogeneity.

The above results can be collected together in the following theorems:

Theorem 1 *The necessary and sufficient conditions for a Riemann-Cartan Gödel-type space-time to be ST (locally) homogeneous are those given by equations (3.20) – (3.22).*

Theorem 2 *All ST locally homogeneous Riemann-Cartan Gödel-type space-times admit a five-dimensional group of affine-isometric motion and are characterized by three independent parameters ℓ , m^2 and ω : identical triads (ℓ, m^2, ω) specify equivalent space-times.*

As the parameter ω is known to be essentially the rotation in Gödel-type space-times, a question which naturally arises here is whether there is any simple geometrical interpretation for the parameters ℓ and m . As the parameter ℓ is a measure of the strength of the

torsion it clearly has the geometrical interpretation usually associated with the torsion tensor. We have not been able to figure out a simple geometrical interpretation for the parameter m , though. The parameter m^2 , nevertheless, has been used to group the general class of Gödel-type metrics into three disjoint subclasses, namely: (i) the hyperbolic class ($m^2 > 0$), (ii) the circular class ($m^2 \equiv -\mu^2 < 0$), and the linear class, where $m^2 = 0$ (see in this regard [19]).

It is worth emphasizing that when $\ell = 0$ eqs. (3.23) – (3.38) reduce to the corresponding equations for Riemannian Gödel-type space-times (eqs. (3.12) – (3.15) and (3.18) – (3.21) in [21]). Therefore, the results in [21] can be reobtained as a special case of our study here. So, for example, the above theorems 1 and 2 generalize the corresponding theorems in [21] (theorems 1 and 2 on page 891).

It should be noticed that the Riemannian ST-homogeneous Gödel-type space-times can have a group of isometries of dimension higher than five, as, e.g., when $m^2 = 4\omega^2$ which permits a G_7 , and $\omega = 0$, $m \neq 0$ which allows a G_6 . However, for Riemann-Cartan Gödel-type space-times, apart from the rather special case of flat Riemann-Cartan space-time ($\ell = m = \omega = 0$), there are no relations among the relevant parameters (ℓ, m^2, ω) for which the dimension of the group of affine-isometric motions is higher than five.

As far as the algebraic classification of the nonvanishing Weyl-type and Ricci-type spinors is concerned, from eqs. (3.23) – (3.27) we find that for a general triad (ℓ, m^2, ω) both Weyl-type spinors Ψ_A and ψ_A are Petrov type D, whereas the Ricci-type spinors $\Phi_{AB'}$ and $\phi_{AB'}$ are both of Segre type [1,1(11)]. There exist, nevertheless, many instances for which these algebraic types can be more specialized. We mention a few:

1. When either $m = \ell/3 = 2\omega$ or $m^2 = \ell^2/2$, $\omega = 0$, Ψ_A and ψ_A are, respectively, Petrov 0 and D, while both $\Phi_{AB'}$ are Segre type [(1,11)1]; and both $\phi_{AB'}$ are type [1,1(11)];
2. For $\ell = 2\omega$ and $m \neq 0$, Ψ_A is Petrov D, ψ_A is Petrov 0, $\Phi_{AB'}$ is type [(1,1)(11)],

and $\phi_{AB'}$ is Segre type 0;

3. When $\ell = 2\omega$ and $m = 0$, both Ψ_A and ψ_A are Petrov 0, and $\Phi_{AB'}$ and $\phi_{AB'}$ are both Segre type 0;
4. For $m = \omega = 0$, $\ell \neq 0$ (Riemannian flat space-time), Ψ_A and ψ_A are both Petrov type D, while $\Phi_{AB'}$ is Segre type $[1,(111)]$; and $\phi_{AB'}$ is type $[1,1(11)]$.

Regarding the classification of the nonvanishing parts of the torsion spinors one can easily find that for $\ell \neq 0$ the spinor $\mathcal{P}_{AX'}$ corresponds to a space-like vector, with $SO(2, 1)$ as its group of isotropy. The Lanczos spinor $\mathcal{L}_{ABCX'}$ is invariant under the spatial rotation (3.32) (one-dimensional isotropy group).

It should be noticed that the equivalence problem techniques, as formulated in Ref. [13] and embodied in the suite of computer algebra programs TCLASSI which we have used in this work, can certainly be used in more general contexts, as for example in the examination of Riemann-Cartan Gödel-type family space-times in which the torsion, although polarized along the direction of the rotation, does not share the translational symmetries of the metric [29]. We have chosen the case of the present article because it gives a simple illustration of our approach to the equivalence problem techniques applied to Einstein-Cartan Gödel-type solutions which have already been discussed in the literature (see Ref. [30] and references therein quoted on Gödel-type solutions with torsion).

As well as specialist systems such as SHEEP, on which TCLASSI is based, all the main general-purpose computer algebra systems have some sort of facilities for calculation in general relativity. Indeed, extensive sets of programs useful in GR are available with REDUCE, MAPLE and MACSYMA, and with MATHEMATICA through the MATHTensor package. By contrast, as far as we are aware, the existing facilities in computer algebra systems for calculations in theories with non-zero torsion are quite limited. Actually, we only know of the REDUCE programs for applications to Poincaré gauge field theory written by J. Dermott McCrea in collaboration with F. W. Hehl [33] and a set of MATHEMATICA

programs for calculation in RC manifolds written by H. H. Soleng and called CARTAN [31] (see also [32]). McCrea’s programs are written using the REDUCE package EXCALC [33, 34]. These programs, however, do not contain the implementation of the equivalence problem for Riemann-Cartan manifolds. The LISP-based system TCLASSI was devised with the equivalence problem of RC manifold in mind, and is so far the only package that incorporates the equivalence problem techniques (see also [14] and [16]). Furthermore, also in TTG there is room for specialized systems like TCLASSI. The major reason for this is that they tend to be more efficient than general-purpose systems. For a comparison of CPU times for a specific metric in GR, for example, see MacCallum [11, 35].

To conclude, we should like to emphasize that as no field equations were used to show the above results, they are valid for every Riemann-Cartan Gödel-type solution regardless of the torsion theory of gravitation one is concerned with, in particular they hold for the Riemann-Cartan Gödel-type class of solutions discussed in [30] and [36], which were found in the context of Einstein-Cartan theory.

Acknowledgments

J.B. Fonseca-Neto and M.J. Rebouças gratefully acknowledge financial assistance from CNPq.

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